

**TWO-PARAMETER DEFORMATION OF THE POINCARÉ
ALGEBRA**

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ABSTRACT. We examine a two-parameter (\hbar, λ) deformation of the Poincaré algebra which is covariant under the action of $SL_q(2, C)$. When $\lambda \rightarrow 0$ it yields the Poincaré algebra, while in the $\hbar \rightarrow 0$ limit we recover the classical quadratic algebra discussed previously in [1], [2]. The analogues of the Pauli-Lubanski vector w and Casimirs p^2 and w^2 are found and a set of mutually commuting operators is constructed.

1. INTRODUCTION

In [1], [2] we proposed a two-parameter deformation of Poincaré algebra which transforms covariantly under the action of $SL_q(2, C)$. The algebra appears to be distinct from systems discussed previously, e.g. in [3], [4], [5], [6], [7], [8], [9], [10], [11] and it has the advantage that it can be expressed very compactly in terms of two 2×2 matrices which we denote by P and Γ . The corresponding matrix elements P_{ij} and Γ_{ij} are operators analogous to momentum and angular momentum, respectively. In order to have a ten dimensional algebra, we imposed the following constraints on P and Γ : P is hermitian (with respect to some anti-involution operation \dagger), while Γ satisfies a certain deformed unimodularity condition which we define shortly.

We shall examine this deformed Poincaré algebra in more detail in this article. In particular, we shall be interested in obtaining its Casimir operators, along with a complete set of commuting operators. Representations of this algebra will be constructed in forthcoming paper.

The algebra of [2] is given by:

$$(1.1) \quad R P R^{-1} P = P R^{-1} P R,$$

$$(1.2) \quad R^{-1} \Gamma R \Gamma = \Gamma R \Gamma R^{-1},$$

$$(1.3) \quad R \Gamma R^{-1} \bar{\Gamma} = \bar{\Gamma} R \Gamma R^{-1},$$

$$(1.4) \quad R^{-1} P R \Gamma = \Gamma R^{-1} P R^{-1},$$

where $\bar{\Gamma} = \Gamma^{\dagger-1}$. We use tensor product notation, labels 1 and 2 denote different vector spaces, $P_1 = P \otimes \mathbb{I}$, $P_2 = \mathbb{I} \otimes P$, etc. where \mathbb{I} is 2×2 unit matrix. The R -

matrix is given by

$$(1.5) \quad R_{12} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

and satisfies the quantum Yang-Baxter equation. Additional commutational relations are obtained by hermitian conjugation. Here q is a real number.

The $P-P$ commutational relations (1.1) are already known, see for instance [12], [13]. They are consistent with P being hermitian. The commutational relations (1.2-1.4) are new. We supplement them by imposing a deformed unimodularity condition on Γ which we now explain. By expanding (1.1-1.4) in terms of matrix elements it can be shown that the following quadratic combinations of Γ 's and $\bar{\Gamma}$'s are in the center of the algebra and we can therefore make them equal to 1:

$$(1.6) \quad \det_{\frac{1}{q}}(\Gamma^T) = \Gamma_{11}\Gamma_{22} - q^2\Gamma_{21}\Gamma_{12} = 1,$$

$$(1.7) \quad \det_q(\bar{\Gamma}) = \bar{\Gamma}_{11}\bar{\Gamma}_{22} - \frac{1}{q^2}\bar{\Gamma}_{12}\bar{\Gamma}_{21} = 1.$$

(1.6-1.7) are not independent: one can be obtained from the other by applying hermitian conjugation and using the expressions for Γ^{-1} and $\bar{\Gamma}^{-1}$

$$(1.8) \quad \bar{\Gamma}^{-1} = \frac{\begin{pmatrix} \bar{\Gamma}_{22} & -\frac{1}{q^2}\bar{\Gamma}_{12} \\ -\frac{1}{q^2}\bar{\Gamma}_{21} & \bar{\Gamma}_{11} + (q^2 - 1)\bar{\Gamma}_{22} \end{pmatrix}}{\det_q(\bar{\Gamma})},$$

$$(1.9) \quad \Gamma^{-1} = \frac{\begin{pmatrix} q^2\Gamma_{22} - (q^2 - 1)\Gamma_{11} & -q^2\Gamma_{12} \\ -q^2\Gamma_{21} & \Gamma_{11} \end{pmatrix}}{\det_{\frac{1}{q}}(\Gamma^T)}$$

and (1.2).

By expressing q in terms of two parameters λ and \hbar according to $q = e^{\hbar\lambda}$ we can define two distinct limits: $\hbar \rightarrow 0$ which we refer to as the "classical limit" and $\lambda \rightarrow 0$ which we call "canonical limit". It is the **latter** and not the former that leads to Poincaré algebra. For this to happen we will need to make Γ depend on λ in a certain way as we explain in Sec.3.

The outline of this paper is as follows. In Sec.2, 3 we examine the limits $\hbar \rightarrow 0$ and $\lambda \rightarrow 0$, respectively. The covariance of the algebra (1.1-1.4) under $SL_q(2, C)$ transformations is shown in Sec. 4. The analogue of the Pauli-Lubanski vector is discussed in Sec. 5. We use it along with P to construct two Casimir operators in Sec. 6. These Casimir operators are invariant under the action of the quantum Lorentz group. In Sec. 7 we find a set of commuting operators which can be used to construct the representation of our algebra. Some concluding remarks are given in Sec. 8.

2. CLASSICAL LIMIT

We begin with the classical limit. Since the only dependence on \hbar comes from R -matrix, we need to know its classical limit¹

$$(2.1) \quad R_{12} \xrightarrow{\hbar \rightarrow 0} \mathbb{I} + (-i\hbar)r + \frac{(-i\hbar)^2}{2}r^2 + O(\hbar^3),$$

where r is the classical r -matrix

$$(2.2) \quad r = \frac{i\lambda}{2} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & 4 & -1 & \\ & & & 1 \end{pmatrix}$$

which satisfies the classical Yang-Baxter equation. Then the algebra (1.1-1.4) becomes²

$$(2.3) \quad \left\{ p, p \right\}_{1 \ 2} = \left(r p p + p p r^\dagger - p r^\dagger p - p r p \right)_{1 \ 2},$$

$$(2.4) \quad \left\{ \gamma, \gamma \right\}_{1 \ 2} = \left(r^\dagger \gamma \gamma + \gamma \gamma r - \gamma r \gamma - \gamma r^\dagger \gamma \right)_{1 \ 2},$$

$$(2.5) \quad \left\{ \gamma, \bar{\gamma} \right\}_{1 \ 2} = \left(r \gamma \bar{\gamma} + \gamma \bar{\gamma} r - \bar{\gamma} r \gamma - \gamma r \bar{\gamma} \right)_{1 \ 2},$$

$$(2.6) \quad \left\{ p, \gamma \right\}_{1 \ 2} = \left(r^\dagger p \gamma + p \gamma r - \gamma r^\dagger p - p r^\dagger \gamma \right)_{1 \ 2},$$

where $\{, \}$ denotes a Poisson bracket which is obtained from a commutator as follows:

$$\frac{[,] }{i\hbar} \xrightarrow{\hbar \rightarrow 0} \{, \}.$$

Obviously, this is not Poincaré algebra. It is instead the Poisson bracket algebra discussed in [1], [2]. It was shown to be consistent with Jacobi identities, hermitian conjugation, as well as being Lie-Poisson with respect to Lorentz transformations.

3. CANONICAL LIMIT

We next discuss the canonical limit. Now we write $\Gamma = e^{i\lambda J}$ and $\bar{\Gamma} = e^{i\lambda J^\dagger}$. Then in the limit $\lambda \rightarrow 0$:

$$\left[P, P \right]_{1 \ 2} = 0, \quad \left[J, J^\dagger \right]_{1 \ 2} = 0, \quad \left[J, J \right]_{1 \ 2} = 2i\hbar\Pi \left(J - J \right)_{1 \ 2}, \quad \left[P, J \right]_{1 \ 2} = i\hbar P_1 (2\Pi - \mathbb{I}),$$

where

$$\Pi = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

¹Although it looks like $R = e^{-i\hbar r}$, it is not so. Starting from the third order in \hbar this equality is violated but only in 3-2 entry of the matrix.

²From now on we use small letters for classical variables and capital letters for quantum operators.

is the permutation operator $J_1 \Pi = \Pi J_2$, $\Pi J_1 = J_2 \Pi$. The above algebra is the Poincaré algebra. To see this we need only express

$$(3.1) \quad P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = -\mathbb{I}P_0 + \sigma_k P_k = \begin{pmatrix} -P_0 + P_3 & P_1 - i P_2 \\ P_1 + i P_2 & -P_0 - P_3 \end{pmatrix},$$

$$(3.2) \quad J = \sigma_k \left(J_{k0} - \frac{i}{2} \epsilon_{kmn} J_{mn} \right) = \begin{pmatrix} -iJ_{12} + J_{30} & -iJ_{23} - iJ_{20} - J_{31} + J_{10} \\ -iJ_{23} + iJ_{20} + J_{31} + J_{10} & iJ_{12} - J_{30} \end{pmatrix},$$

σ_k being Pauli matrices, and then we arrive at the usual form:

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \quad [P_\mu, J_{\nu\rho}] = i\hbar (\eta_{\mu\rho} P_\nu - \eta_{\mu\nu} P_\rho), \\ [J_{\mu\nu}, J_{\rho\sigma}] &= i\hbar (\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\rho\nu} + \eta_{\nu\rho} J_{\sigma\mu}), \\ \eta &= \text{diag}(-1, 1, 1, 1). \end{aligned}$$

Thus the algebra (1.1-1.4) is a two-parameter deformation of the Poincaré algebra.

4. $SL_q(2, C)$ COVARIANCE

It remains to show that (1.1-1.4) is covariant with respect to $SL_q(2, C)$ transformations. A $SL_q(2, C)$ matrix T is defined using the commutational relations³:

$$(4.1) \quad \begin{matrix} R & T & T \\ 12 & 1 & 2 \end{matrix} = \begin{matrix} T & T & R \\ 2 & 1 & 12 \end{matrix}$$

which are well known (for review see e.g. [14]). The action of $SL_q(2, C)$ on the operators P and Γ involves T as well as its hermitian conjugate $T^\dagger = \bar{T}^{-1}$. We must then specify the commutational relations involving \bar{T} :

$$(4.2) \quad \begin{matrix} R & T & \bar{T} \\ 12 & 1 & 2 \end{matrix} = \begin{matrix} \bar{T} & T & R \\ 2 & 1 & 12 \end{matrix},$$

$$(4.3) \quad \begin{matrix} R & \bar{T} & \bar{T} \\ 12 & 1 & 2 \end{matrix} = \begin{matrix} \bar{T} & \bar{T} & R \\ 2 & 1 & 12 \end{matrix}.$$

Since the "deformed" determinants of T and \bar{T} commute with everything we can set them equal to one:

$$(4.4) \quad \det_{\frac{1}{\sqrt{q}}}(T) = T_{11}T_{22} - qT_{12}T_{21} = 1,$$

$$(4.5) \quad \det_{\frac{1}{\sqrt{q}}}(\bar{T}) = \bar{T}_{11}\bar{T}_{22} - q\bar{T}_{12}\bar{T}_{21} = 1.$$

³In the canonical limit the matrix elements of T , of course, commute. In the classical limit, however, we get non-trivial Poisson brackets for the classical variables t associated with T :

$$\left\{ t, \bar{t} \right\} = \left[r, \frac{t\bar{t}}{12} \right]$$

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$$\left\{ \bar{t}, \bar{t} \right\} = \left[r, \frac{\bar{t}\bar{t}}{12} \right]$$

here $t, \bar{t} \in SL(2, C)$, $\bar{t} = t^{\dagger-1}$

Just like (1.6-1.7), (4.4-4.5) are not independent and one can be obtained from the other by hermitian conjugation with the help of

$$(4.6) \quad T^{-1} = \frac{\begin{pmatrix} T_{22} & -\frac{1}{q}T_{12} \\ -qT_{21} & T_{11} \end{pmatrix}}{\det \frac{1}{\sqrt{q}}(T)},$$

$$(4.7) \quad \bar{T}^{-1} = \frac{\begin{pmatrix} \bar{T}_{22} & -\frac{1}{q}\bar{T}_{12} \\ -q\bar{T}_{21} & \bar{T}_{11} \end{pmatrix}}{\det \frac{1}{\sqrt{q}}(\bar{T})}.$$

Under the action of $SL_q(2, C)$ the operators P , Γ and $\bar{\Gamma}$ transform according to

$$(4.8) \quad \begin{aligned} P &\rightarrow P' = \bar{T}PT^{-1}, \\ \Gamma &\rightarrow \Gamma' = T\Gamma T^{-1}, \\ \bar{\Gamma} &\rightarrow \bar{\Gamma}' = \bar{T}\bar{\Gamma}\bar{T}^{-1}. \end{aligned}$$

The first relation states that P transforms as a vector, while the latter two define the adjoint action for $SL_q(2, C)$. Assuming, as usual, that the the matrix elements of $(P, \Gamma, \bar{\Gamma})$ commute with those of (T, \bar{T}) it is not hard to show that (1.1-1.4) are covariant under (4.8). For example,

$$\begin{aligned} RPR^{-1}P &\rightarrow RPR^{-1}P' = \bar{T}\bar{T}RPR^{-1}PT^{-1}T^{-1} = \\ &\bar{T}\bar{T}PR^{-1}PRT^{-1}T^{-1} = P'R^{-1}P'R. \end{aligned}$$

5. PAULI-LUBANSKI VECTOR

We obtained the following classical deformation of Pauli-Lubanski vector in [1], [2]:

$$(5.1) \quad w = \frac{1}{2\lambda} (\bar{\gamma}^{-1}p\gamma - p) = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \begin{pmatrix} -w_0 + w_3 & w_1 - i w_2 \\ w_1 + i w_2 & -w_0 - w_3 \end{pmatrix}$$

which transforms as a Lorentz vector and reduces to the standard Pauli-Lubanski vector when $\lambda \rightarrow 0$. The Poisson brackets for w are:

$$(5.2) \quad \left\{ w, \gamma \right\} = r^\dagger w \gamma + w \gamma r - \gamma r^\dagger w - w r^\dagger \gamma,$$

$$(5.3) \quad \left\{ w, \bar{\gamma} \right\} = r w \bar{\gamma} + w \bar{\gamma} r - \bar{\gamma} r^\dagger w - w r \bar{\gamma},$$

$$(5.4) \quad \left\{ w, p \right\} = r w p + w p r^\dagger - p r^\dagger w - w r p,$$

$$(5.5) \quad \left\{ w, w \right\} = r w w + w w r^\dagger - w r^\dagger w - w r w - i\Pi \left(w p - w p \right).$$

Because of the last term in the last Poisson bracket the quantum algebra for w will depend on λ not just through $q = e^{\hbar\lambda}$ but also separately.

Let us define the quantum analog of w according to:

$$(5.6) \quad W = a \left(\beta \bar{\Gamma}^{-1} P \Gamma - P \right),$$

where $W^\dagger = W$, $a \xrightarrow{\hbar \rightarrow 0} \frac{1}{2\lambda}$, $\beta \xrightarrow{\hbar \rightarrow 0} 1$. Using (5.6) and (1.1-1.4) we get:

$$(5.7) \quad \begin{matrix} R^{-1} & W & R & \Gamma \\ 21 & 1 & 21 & 2 \end{matrix} = \begin{matrix} \Gamma & R^{-1} & W & R^{-1} \\ 221 & 1 & 12 & \end{matrix},$$

$$(5.8) \quad \begin{matrix} R & W & R^{-1} & \bar{\Gamma} \\ 12 & 1 & 12 & 2 \end{matrix} = \begin{matrix} \bar{\Gamma} & R^{-1} & W & R^{-1} \\ 221 & 1 & 12 & \end{matrix},$$

$$(5.9) \quad \begin{matrix} R & W & R^{-1} & P \\ 12 & 1 & 12 & 2 \end{matrix} = \begin{matrix} P & R^{-1} & W & R \\ 221 & 1 & 21 & \end{matrix},$$

$$(5.10) \quad \begin{matrix} R & (W + aP) & R^{-1} & W \\ 12 & 1 & 12 & 2 \end{matrix} = \begin{matrix} W & R^{-1} & (W + aP) & R \\ 2 & 21 & 1 & 21 \end{matrix}.$$

(5.7-5.10) are covariant with respect to (4.8) and W transforms as an $SL_q(2, C)$ vector:

$$(5.11) \quad W \rightarrow W' = \bar{T}WT^{-1}.$$

Notice that in order to have correct classical limit $\hbar \rightarrow 0$ we have to include not only q , but also λ (through a) in the commutational relations. Also note that (5.7-5.10) are satisfied identically for any β from (5.6).

To recover the usual Pauli-Lubanski vector we assume that the canonical limit of a and β is identical to the classical limit, i.e. $a \xrightarrow{\lambda \rightarrow 0} \frac{1}{2\lambda}$, $\beta \xrightarrow{\lambda \rightarrow 0} 1$. Then using $\Gamma = e^{i\lambda J}$ and $\bar{\Gamma} = e^{i\lambda J^\dagger}$ we get $W = -\mathbb{I}W_0 + \sigma_k W_k$, where

$$W_0 = -\frac{1}{2}\epsilon_{kmn}P_kJ_{mn}, \quad W_k = -\frac{1}{2}\epsilon_{kmn}P_0J_{mn} - \epsilon_{lmk}P_lJ_{m0}.$$

In 4-vector notations it can be rewritten in the form:

$$W_\beta = -\frac{1}{2}\epsilon_\beta^{\mu\nu\rho}J_{\mu\nu}P_\rho.$$

6. CASIMIRS

Like with the Poincaré algebra, we can construct two Casimir operators. Both are invariant under $SL_q(2, C)$ transformations as we shall make evident. The following quadratic combination is an analogue of P^2 and is one such Casimir:

$$(6.1) \quad \mathcal{C}_1 = -\det_q(P) = -\langle P, P \rangle_q = (P, P)_q = -P_0^2 + \frac{P_1^2}{q^2} + \frac{P_2^2}{q^2} + P_3^2 \xrightarrow{q \rightarrow 1} P^2,$$

where

$$(6.2) \quad \langle A, B \rangle_q = A_{11}B_{22} - \frac{1}{q^2}A_{12}B_{21}$$

is a combination of matrix elements which often occurs in this algebra so we reserve a special notation for it.

$$(6.3) \quad (A, B)_q = -\frac{1}{q^2 + 1}Tr_q(A\tilde{B})$$

is a deformed scalar product of 4-vectors A and B represented as 2×2 hermitian matrices. \tilde{B} is a deformed adjugate of B -matrix. By this we mean that \tilde{B} has the properties:

$$\begin{aligned} B\tilde{B} &= \tilde{B}B \sim \mathbb{I}, \\ \tilde{B} \xrightarrow{q \rightarrow 1} \tilde{b} &= \begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix}, \quad \tilde{b}b = b\tilde{b} = \det(b)\mathbb{I}. \end{aligned}$$

Tr_q is a deformed trace defined by

$$Tr_q(A) = A_{11} + q^2 A_{22}.$$

The adjugate of P has the following form:

$$(6.4) \quad \tilde{P} = \begin{pmatrix} P_{22} & -\frac{1}{q^2}P_{12} \\ -\frac{1}{q^2}P_{21} & \frac{1}{q^2}(P_{11} + (q^2 - 1)P_{22}) \end{pmatrix}.$$

It was proved directly using Mathematica that \tilde{P} has the same transformation properties⁴ as \bar{P} :

$$\tilde{P}' \rightarrow T\tilde{P}\bar{T}^{-1}.$$

We found another Casimir corresponding to the square of the Pauli-Lubanski vector. It can be expressed in different ways:

$$(6.5) \quad \begin{aligned} \mathcal{C}_{2a} &= a(<P, W>_q + q^2 <W, P>_q) = \\ &= aTr_q(W\tilde{P}) = aTr_q(\tilde{P}W) = -a(q^2 + 1)(W, P)_q, \end{aligned}$$

$$(6.6) \quad \mathcal{C}_{2b} = aTr_q(P\tilde{W}) = aTr_q(\tilde{W}P) = -a(q^2 + 1)(P, W)_q,$$

$$(6.7) \quad \mathcal{C}_{2c} = -<W, W - a(q^2 - 1)P>_q = (W, W)_q,$$

Here⁵

$$(6.8) \quad \tilde{W} = \begin{pmatrix} W_{22} & -\frac{W_{12}}{q^2} \\ -\frac{W_{21}}{q^2} & \frac{W_{11} + (q^2 - 1)W_{22}}{q^2} \end{pmatrix} - a(q^2 - 1)\tilde{P}$$

is an adjugate matrix of W and direct calculations show that \tilde{W} transforms like \bar{W} :

$$\tilde{W}' \rightarrow T\tilde{W}\bar{T}^{-1}.$$

One can check using (5.6) that \mathcal{C}_{2a} , \mathcal{C}_{2b} and \mathcal{C}_{2c} are in fact equal to each other up to \mathcal{C}_1 :

$$\mathcal{C}_{2a} = \mathcal{C}_{2b} - a^2(q^4 - 1)\mathcal{C}_1 = \mathcal{C}_{2c} + \frac{a^2(q^6 - \beta^2)}{q^6}\mathcal{C}_1.$$

\mathcal{C}_{2a} and \mathcal{C}_{2b} have the same classical limit, namely $c_{2a} = -\frac{(p, w)}{\lambda}$, where $(,) = (,)_{q=1}$. The classical limit of \mathcal{C}_{2c} is $c_{2c} = w^2$. At first glance one might think that $c_{2a} \neq c_{2c}$ but it is easy to prove that c_{2a} and c_{2c} are really equal to each other. For this we write

$$w = \frac{1}{2\lambda}(b - p),$$

⁴When $\det_q(P) \neq 0$

$$\tilde{P} = \det_q(P)P^{-1}.$$

Notice though that \tilde{P} exists even when P^{-1} does not, that is for 0-length 4-vectors. To find \tilde{P} we used an ansatz where each element of the matrix is a general linear combination of all P_{ij} . \tilde{P} is uniquely determined up to a factor which goes to 1 when $q \rightarrow 1$.

⁵Again, when $<W, W - a(q^2 - 1)P>_q \neq 0$

$$\tilde{W} = <W, W - a(q^2 - 1)P>_q W^{-1}.$$

To find \tilde{W} we used an ansatz where each element of the matrix is a general linear combination of all W_{ij} and P_{ij} . It was uniquely determined up to a factor which goes to 1 when $q \rightarrow 1$.

where $b = \bar{\gamma}^{-1}p\gamma$. Then $\det(b) = \det(p)$ because $\det(\gamma) = \det(\bar{\gamma}) = 1$. Furthermore

$$2\lambda(p, w) = (p, b) + \det(p)$$

using $(p, p) = -\det(p)$, and thus

$$w^2 = \frac{1}{4\lambda^2}(-\det(b) - \det(p) - (b, p) - (p, b)) = \frac{1}{2\lambda^2}(-\det(p) - (p, b)) = -\frac{(p, w)}{\lambda}.$$

Let us show that our deformed scalar product is really invariant and hence that \mathcal{C}_1 and \mathcal{C}_2 are invariant. Suppose that under $SL_q(2, C)$ transformations

$$A' \rightarrow \bar{T}AT^{-1}, \quad \tilde{B}' \rightarrow T\tilde{B}\bar{T}^{-1}$$

and matrix elements of T commute with A and \tilde{B} . Then

$$\begin{aligned} Tr_q(A'\tilde{B}') &= Tr_q(\bar{T}A\tilde{B}\bar{T}^{-1}) = (\bar{T}_{1k}\bar{T}_{n1}^{-1} + q^2\bar{T}_{2k}\bar{T}_{n2}^{-1}) A_{km}\tilde{B}_{mn} = \\ &= \delta_{kn}^q A_{km}\tilde{B}_{mn} = Tr_q(A\tilde{B}), \quad \text{where} \quad \delta^q = \begin{pmatrix} 1 & 0 \\ 0 & q^2 \end{pmatrix}. \end{aligned}$$

Notice that in all cases discussed above $Tr_q(A\tilde{B}) = Tr_q(\tilde{B}A)$ which is rather unexpected for such a deformed algebra. On the other hand, $Tr_q(A\tilde{B}) \neq Tr_q(B\tilde{A})$, that is the scalar product is not symmetric in general.

7. COMPLETE SET OF MUTUALLY COMMUTING OPERATORS

In addition to the Casimirs the following operators can be included into a complete set of mutually commuting operators necessary to construct a representation:

$$(7.1) \quad \mathcal{K}_1 = Tr_q(P) = P_{11} + q^2 P_{22},$$

$$(7.2) \quad \mathcal{K}_2 = Tr_q(W) = W_{11} + q^2 W_{22} = a(qTr_q(P\Omega) - \mathcal{K}_1),$$

$$(7.3) \quad \mathcal{K}_3 = P_{11} \quad \text{or} \quad P_{22},$$

$$(7.4) \quad \mathcal{K}_4 = \langle \Gamma, \bar{\Gamma} \rangle_q = \Omega_{11},$$

where

$$(7.5) \quad \Omega = \Gamma\bar{\Gamma}^{-1} = \Gamma\Gamma^\dagger = \Omega^\dagger.$$

When constructing a representation, it is more convenient to choose $\mathcal{K}_3 = P_{22}$ because it has simpler commutational relations with other operators.

Ω has very peculiar property: it has the same commutational relations with all matrices $Z = P, \Gamma, \bar{\Gamma}, W$ or Ω :

$$(7.6) \quad \begin{matrix} Z & R & \Omega & R \\ 1 & 21 & 2 & 12 \end{matrix} = \begin{matrix} R & \Omega & R & Z \\ 21 & 2 & 12 & 1 \end{matrix}.$$

It follows from here, by the way, that components of Ω generate a universal enveloping algebra.

In canonical limit \mathcal{K}_1 and \mathcal{K}_2 go to $-2P_0$ and $-2W_0$, respectively, while

$$(7.7) \quad \frac{\Omega - \mathbb{I}}{\lambda} \xrightarrow{\lambda \rightarrow 0} \begin{pmatrix} J_{12} & J_{23} - iJ_{31} \\ J_{23} + iJ_{31} & -J_{12} \end{pmatrix}$$

is a traceless hermitian matrix corresponding to space components of angular momentum. Therefore \mathcal{K}_4 is associated with the component of the angular momentum along the third axis J_{12} .

It was checked that no other linear or quadratic combination of P , W , Γ and $\bar{\Gamma}$ can be included in this set.

Another possible set of mutually commuting operators can be obtained if we *replace* \mathcal{K}_3 by

$$(7.8) \quad \mathcal{K}_5 = Tr(\Omega) \xrightarrow{\lambda \rightarrow 0} 2(1 + 2\lambda^2(J_{12}^2 + J_{23}^2 + J_{31}^2)) + o(\lambda^2)$$

which corresponds to the square of 3-vector of angular momentum (*replace*, because no linear combination of P_{11} and P_{22} , except for \mathcal{K}_1 , commutes with \mathcal{K}_5). Therefore in the set we can have either an analogue of P_3 or \vec{J}^2 but not both⁶.

For the sake of space we do not write out all commutational relations between \mathcal{K}_i and elements of the algebra but only some simple ones:

$$\begin{aligned} [\mathcal{K}_1, P_{ij}] &= 0, \quad [\mathcal{K}_2, P_{ij}] = 0, \\ [\mathcal{K}_4, Z_{11}] &= [\mathcal{K}_4, Z_{22}] = 0, \quad \mathcal{K}_4 Z_{12} = \frac{1}{q^2} Z_{12} \mathcal{K}_4, \quad \mathcal{K}_4 Z_{21} = q^2 Z_{21} \mathcal{K}_4, \end{aligned}$$

where Z can be: P , W , Γ , $\bar{\Gamma}$ or Ω . We shall give a complete set of commutational relations along with representations of the algebra in a forthcoming article.

8. CONCLUSION

Here we remark on possible extension of our algebra. In [1] we deformed the canonical Poisson brackets for a relativistic particle so that the $SL(2, C)$ Poisson-Lie group with Poisson structure given in footnote³ had a Poisson action on the classical observables. The most non-trivial part of [1] was to deform the canonical Poisson brackets between momenta and coordinates $\{x_\mu, p_\nu\} = \eta_{\mu\nu}$. We obtained

$$(8.1) \quad \left\{ \begin{matrix} x \\ 1 \\ 2 \end{matrix}, \begin{matrix} p \\ 1 \\ 2 \end{matrix} \right\} = r x p + x p r^\dagger - p r x - x r^\dagger p - \Pi(f_1^\dagger + f_2)$$

where $f = \exp(i \sin^{-1}(\lambda x p))$. (8.1) was shown to be covariant with respect to $x \rightarrow t x \bar{t}^{-1}$, $p \rightarrow \bar{t} p t^{-1}$. It would be interesting to extend quantum algebra (1.1-1.4) to include the space-time operator X . Upon choosing the Hamiltonian to be equal to \mathcal{C}_1 we could then write down the corresponding Klein-Gordon equation giving the dynamics of a "free" particle, the classical analogue of which was discussed in [1], [2].

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⁶At first glance one might think that $\det(\Omega)$ can also be included in either of two sets but it turned out that it is not independent:

$$\det(\Omega) = \frac{1}{q^2} (1 + (q^2 - 1) \mathcal{K}_4^2).$$

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